Quantum Computation Beyond the "Standard Circuit Model" *

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Abstract

Construction of explicit quantum circuits follows the notion of the "standard circuit model" introduced in the solid and profound analysis of elementary gates providing quantum computation. Nevertheless the model is not always optimal (e.g. concerning the number of computational steps) and it neglects physical systems which cannot follow the "standard circuit model" analysis. We propose a computational scheme which overcomes the notion of the transposition from classical circuits providing a computation scheme with the least possible number of Hamiltonians in order to minimize the physical resources needed to perform quantum computation and to succeed a minimization of the computational procedure (minimizing the number of computational steps needed to perform an arbitrary unitary transformation). It is a general scheme of construction, independent of the specific system used for the implementation of the quantum computer. The open problem of controllability in Lie groups is directly related and rises to prominence in an effort to perform universal quantum computation.

Keywords. Quantum Gates, Quantum Computation, Quantum Control Theory.

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1 The "Standard Circuit Model"

The "standard circuit model" is an established proposal to implement quantum gates in quantum computation [1]. In this model essential is the notion of the universal gate [2]. Thus, any given quantum gate (any given unitary transformation of the quantum system that implements the quantum computer) can be analyzed using a set of basic gates, known as universal gates. The selection of the set of universal gates is not unique [3]. One-qubit gates can be analyzed using only Hadamard and phase gates. Two-qubit gates can be analyzed using Hadamard, phase and the CNOT gate and this is generalized in the case of N-qubit gates, while it was noted that in the general case an infinite number of steps are needed to perform a gate explicitly [4].

In the "standard circuit model", physical systems are neglected if they cannot copy easily the model (if someone cannot perform easily one of the selected universal gates). Also, neither the number of computational steps nor the total time to perform computation are optimal [5].

2 Quantum Control Theory

In quantum control theory, the generalization of the control theory in quantum systems, a system is said to be controllable if an arbitrary Lie group element $W \in SU(2^N)$ can be decomposed in finite time as

$$W = e^{-ia_n J^{(n)}} \dots e^{-ia_2 J^{(2)}} e^{-ia_1 J^{(1)}} \tag{1}$$

where $J^{(k)} \in \{J_1, J_2, \dots, J_m\}$ are generators of the corresponding $su(2^N)$ Lie algebra and $a_i \in \mathbb{R}$. In the case of quantum computation, W is equivalent with an arbitrary unitary transformation (up to a global phase) so it is equivalent with an arbitrary N-qubit gate. $J^{(k)}$ corresponds to the Hamiltonians describing the system under consideration while a_i are equivalent with time parameters t_i. The controllability on Lie groups from a mathematical point of view was studied in [6, 7, 8, 9]. This direct relation between the problem of controllability in Lie groups and the problem of universal quantum computation allows us to approach quantum computation with an alternative way beyond the "standard circuit model". In this approach if the selected Hamiltonians $J_1,\ J_2,\ \dots, J_m$ form a complete set of operators, then every $W \in SU(2^N)$ can be exactly realized using a finite number of steps, although this number of steps is not fixed, where in the case of the "standard circuit model" the same element $SU(2^N)$ could be approximately realized using an infinitely number of steps. The order of generation (the number of computational steps required to perform an arbitrary N-qubit gate) is available for arbitrary Hamiltonians only in the case of the $SU(2)$ group (one-qubit gates) via the Lowental's criterion [10]. In this case, only two Hamiltonians $\{J_1,\ J_2\}$ are sufficient to form a complete set. If the Hamiltonians are orthogonal, i.e. $\mathrm{Trace}(J_1 J_2) = 0$, then three at most steps are required, to realizing any $W \in SU(2)$). When $\mathrm{Trace}(J_1 J_2) \neq 0$, the number of steps are given by the Lowental's criterion, but the algorithm to obtain the solution is not known.

In the case of higher order groups there is an analysis based on the Cartan decomposition of the $su(2^N)$ algebras [8]. This analysis provides also an analytical way of calculating

the corresponding time parameters (Euler angles) in the case of the $SU(4)$ (2-qubit gates). On the same spirit is the proposal for exact computation by Whaley and collaborators [11]. Open problems in Lie groups controllability are:

a) *SU(2) group (one-qubit gates).* An algorithm which, given an arbitrary couple of generators–Hamiltonians, will be able to provide analytically the time parameters to perform universal computation, if the number of required steps are more than three.

b) *Higher order groups.* A criterion for minimum number of steps to generate an arbitrary element of the group (which corresponds to an arbitrary N-qubit gate, respectively) in the case where the generators–Hamiltonians are not orthogonal. Algorithms to evaluate the corresponding time parameters.

3 Quantum Gates Using the Intrinsic Abilities of a Physical System

The main points of our proposal can be summarized briefly as follows

- Instead of forcing a physical system to act as a predetermined set of universal gates we focus on the ability of the physical system to act as a quantum computer using only its natural available interactions (encoded universality [12]).
- Construction of any given gate and algorithm in terms of a minimal configuration and computational procedure.
- Minimized finite number of steps, evolving in time according to a finite number of basic, intrinsic Hamiltonians, controlled by a minimal finite number of classical switches (the selection of the switches is not unique).
- Implementation does not depend on the psecific system used as a qubit. Several solid state proposals as charge Josephson junctions, SQUIDs, quantum dots have been tested but our proposal can be extended to NMR quantum computation, trapped ions etc in order to test it with various physical systems described by different Hamiltonians and interactions.

This computer consists of one cell controlled by external binary switches and evolving in time using these switches. Quantum gates and algorithms are translated into manipulation of these switches. It is a simple device which overcomes the notion of transposition from classical circuits and does not have any "quantum" connections (one of the difficult parts in physical implementation-especially in solid state devices).

The above proposal is based in the following mathematical Propositions:

Proposition 3.1 *A number of $N+1$ switches are sufficient for universal quantum computation in a N-qubit device.*

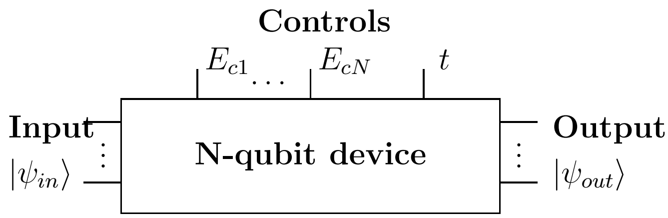


Figure 1: *Abstract N-qubit device*

Proposition 3.2 *A set of $N+2$ Hamiltonians provided by the basic Hamiltonian of the N-qubit device through appropriate tuning of the $N+1$ switches, can generate the $su(2^N)$ algebra.*

Proposition 3.3 *The construction scheme of any quantum gate consists of a finite number of steps evolving in time according to a finite number of basic Hamiltonians (reiterating in cyclic pattern) and provided by proper switches' manipulation.*

$$U = \dots e^{-it_4 H_4} e^{-it_3 H_2} e^{-it_2 H_3} e^{-it_1 H_1} \tag{2}$$

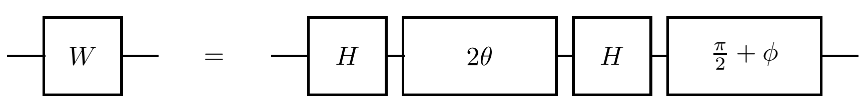
It is an open conjecture that every U can be generated exactly by $\mathcal{O}(4^N)$ steps.

The manipulations of the quantum computer can be codified by a rudimentary *Quantum Machine Language* [13].

4 Results

4.1 One-qubit gates

According to the "standard model" the most usual analysis of an arbitrary one-qubit gate includes two Hadamard gates and two phase gates.



Even if both Hadamard and phase gate can be realized in one computational step, then at least four steps are required to perform universal quantum computation. Usually decompositions of elementary one-qubit gates require at least 8 computational steps according to this analysis and it is performed most of the times in systems which provide us with orthogonal Hamiltonians.

With the presented computational scheme the results in the case of the one-qubit gates are the following:
Orthogonal Hamiltonians If the Hamiltonians are orthogonal, i.e. $(H_1, H_2) = \mathrm{Trace}(H_1 H_2) = 0$, then two Hamiltonians and three computational steps at most are required, to realize any $W \in SU(2)$.

$$W = e^{-it_3 H_1} e^{-it_2 H_2} e^{-it_1 H_1} \tag{3}$$

For example in the case of NMR (where orthogonal Hamiltonians are used) we can perform universal quantum computation within three computational steps, while the analytical solutions for the time parameters are the trivial Euler angles.

Non-Orthogonal Hamiltonians If the Hamiltonians are non-orthogonal i.e. $(H_1, H_2) = \mathrm{Trace}(H_1 H_2) \neq 0$, the number of steps–the order of generation is $n = k + 2$, given by the Lowental's criterion

$$\cos\left(\frac{\pi}{k}\right) < \frac{|(H_1, H_2)|}{(H_1, H_1)^{1/2}(H_2, H_2)^{1/2}} \leq \cos\left(\frac{\pi}{k+1}\right), \quad k \geq 2 \tag{4}$$

and the corresponding construction scheme is the following

$$W = e^{-it_n H_1} \dots e^{-it_3 H_1} e^{-it_2 H_2} e^{-it_1 H_1} \tag{5}$$

For example in the case of the charge Josephson junctions where the general Hamiltonian is $H = \frac{1}{2} E_c \sigma_z - \frac{1}{2} E_J \sigma_x$ manipulation of the bias energy E_c which is controlled by the binary switch of gate voltage V_g, provides the following non-orthogonal Hamiltonians

$$H_1 = -\frac{1}{2} E_J \sigma_x \quad \text{and} \quad H_2 = \frac{1}{2} E_c \sigma_z - \frac{1}{2} E_J \sigma_x \tag{6}$$

The pair $\{H_1, H_2\}$ generates the $su(2)$ algebra but since $\mathrm{Trace}(H_1 H_2) \neq 0$ the whole $SU(2)$ group cannot be cover in 3 steps.
The Lowenthal's parameter ψ is

$$\psi = \frac{|(H_1, H_2)|}{(H_1, H_1)^{1/2}(H_2, H_2)^{1/2}} = \frac{\frac{E_J}{E_c}}{\sqrt{1 + \frac{E_J^2}{E_c^2}}} = \frac{x}{\sqrt{1 + x^2}} \quad \text{where } x = \frac{E_J}{E_c} \tag{7}$$

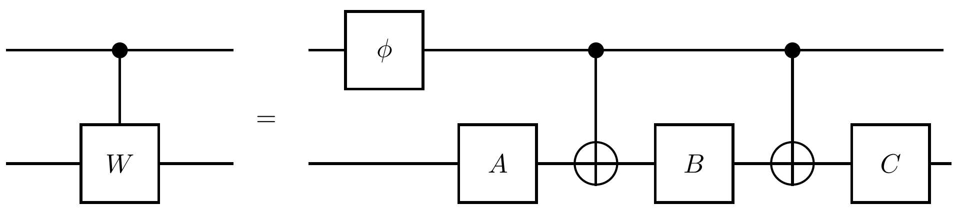
If x is small enough, then $\psi < \cos\frac{\pi}{3}$ and every element of the $SU(2)$ group $W = w_0 \mathbb{I} - i(w_1 \sigma_x + w_2 \sigma_y + w_3 \sigma_z)$ (one-qubit gate), can be constructed in 4 steps at most. The corresponding analytical solutions in that case are

$$t_1 = -\tfrac{2}{E_J} \arctan\left(\tfrac{Ec(w_0 w_3 - w_1 w_2) + \sqrt{w_2^2 + w_3^2}\sqrt{E_c^2(w_0^2 + w_1^2) - E_J^2(w_2^2 + w_3^2)}}{-E_c(w_0 w_2 + w_1 w_3) + E_J(w_2^2 + w_3^2)}\right) + \tfrac{4k_1\pi}{E_J}$$
$$t_2 = -\tfrac{2}{\sqrt{E_c^2 + E_J^2}} \arctan\left(\tfrac{\sqrt{E_c^2 + E_J^2}\sqrt{w_2^2 + w_3^2}}{\sqrt{E_c^2(w_0^2 + w_1^2) - E_J^2(w_2^2 + w_3^2)}}\right) + \tfrac{4k_2\pi}{\sqrt{E_c^2 + E_J^2}}$$
$$t_3 = -\tfrac{2}{E_J} \arctan\left(\tfrac{Ec(w_0 w_3 + w_1 w_2) + \sqrt{w_2^2 + w_3^2}\sqrt{E_c^2(w_0^2 + w_1^2) - E_J^2(w_2^2 + w_3^2)}}{E_c(w_0 w_2 - w_1 w_3) + E_J(w_2^2 + w_3^2)}\right) + \tfrac{4k_3\pi}{E_J}$$

$$
\begin{aligned}
t_4 = \quad & \frac{2}{\sqrt{E_c^2+E_J^2}} \operatorname{arccot}\left(\frac{-2(\mathrm{w}_1\mathrm{w}_2+\mathrm{w}_0\mathrm{w}_3)\sqrt{1+\mathrm{x}^2}}{2(\mathrm{w}_0^2+\mathrm{w}_1^2-(\mathrm{w}_2^2+\mathrm{w}_3^2)\,\mathrm{x}^2)} + \right. \\
& \left. + \frac{\sqrt{4(w_1w_2+w_0w_3)^2(1+x^2)-4(w_0^2+w_1^2-(w_2^2+w_3^2)\,x^2)(2(w_0w_2-w_1w_3)\,x-(w_2^2+w_3^2)(-1+x^2))}}{2(w_0^2+w_1^2-(w_2^2+w_3^2)\,x^2)} \right) + \\
& + \frac{4k_4\pi}{\sqrt{E_c^2+E_J^2}} \quad \text{where } \; k_1, k_2, k_3, k_4 \in \mathbb{N}
\end{aligned}
$$

4.2 Two qubit gates

Analysis according to the "standard circuit model" requires at least 5 Hamiltonians and 19 computational steps and and it is performed most of the times in systems which provides orthogonal Hamiltonians.



In the case of orthogonal Hamiltonians there is the Cartan decomposition of the $SU(2^N)$ group [8], directly applied to the $SU(4)$ group, which gives analytical solutions and was recently extended with an algorithm to realize every $SU(2^N)$ [14]. The decomposition provided by [8] requires 5 different Hamiltonians and 27 computational steps to simulate an arbitrary gate while the number of computational steps reduces to 19 in the case of a controlled gate.

Next we show the results of numerical simulations of the present computational scheme: **Orthogonal Hamiltonians** If the Hamiltonians are orthogonal (e.g. Heisenberg interaction [17], BQHD [18], SQUIDs [15]) then with two binary switches providing us with 3 different Hamiltonians and within 15 computational steps we cover the $SU(4)$ group (conjecture) and all the tested gates are successfully simulated.

For example, in a system described by a general Hamiltonian of the form $H = \sum_i^N \bar{B}^i(t)\hat{\sigma}^{(i)} + \sum_{i\neq j} J_{ab}^{ij}(t)\hat{\sigma}_a^{(i)}\hat{\sigma}_b^{(j)}$ (Heisenberg interaction), only 3 Hamiltonians

$$
\begin{aligned}
H_1 &= B^1\,\sigma_z^{(1)} \\
H_2 &= B^2\,\sigma_x^{(2)} \\
H_3 &= J_{12}\left(\sigma_x^{(1)}\sigma_x^{(2)} + \sigma_y^{(1)}\sigma_y^{(2)} + \sigma_z^{(1)}\sigma_z^{(2)}\right) \qquad (8)
\end{aligned}
$$

are sufficient for universal quantum computation in 15 computational steps

$$U = e^{H_3 t_{15}} e^{H_2 t_{14}} e^{H_1 t_{13}} \ldots e^{H_3 t_3} e^{H_2 t_2} e^{H_1 t_1} \qquad (9)$$

Non-Orthogonal Hamiltonians If the Hamiltonians are non-orthogonal (charge Josephson junctions [15], quantum dots [16], permanent interaction which cannot be switched off

etc) but the interaction between the qubits is weak, then using 4 different Hamiltonians and within 15 computational steps (time parameters) a large part of the $SU(4)$ is covered and all the known important gates for quantum computation are successfully simulated. In general, the weaker the interaction, the larger the part of the group covered (more gates can be simulated).

a) *Permanent Interaction.* If the interaction J_{12} of the previous paradigm can not be switched of then a construction scheme with two binary switches and 3 non-orthogonal Hamiltonians

$$\begin{aligned} H_1 &= B^1\,\sigma_z^{(1)} + J_{12}\,\left(\sigma_x^{(1)}\sigma_x^{(2)} + \sigma_y^{(1)}\sigma_y^{(2)} + \sigma_z^{(1)}\sigma_z^{(2)}\right) \\ H_2 &= B^2\,\sigma_x^{(2)} + J_{12}\,\left(\sigma_x^{(1)}\sigma_x^{(2)} + \sigma_y^{(1)}\sigma_y^{(2)} + \sigma_z^{(1)}\sigma_z^{(2)}\right) \\ H_3 &= J_{12}\,\left(\sigma_x^{(1)}\sigma_x^{(2)} + \sigma_y^{(1)}\sigma_y^{(2)} + \sigma_z^{(1)}\sigma_z^{(2)}\right) \end{aligned} \tag{10}$$

simulates all the basic gates in 15 steps (9).

b) *Charge Josephson junctions.* A system of two identical coupled Josephson junctions is described by the following general Hamiltonian $\frac{1}{2}E_{c_1}\,\sigma_z^{(1)} - \frac{1}{2}E_{J_1}\,\sigma_x^{(1)} + \frac{1}{2}E_{c_2}\,\sigma_z^{(2)} - \frac{1}{2}E_{J_2}\,\sigma_x^{(2)} - \frac{1}{2}E_L\,\sigma_y^{(1)}\,\sigma_y^{(2)}$. Manipulation of 3 binary switches of the system provides the following 4 non-orthogonal Hamiltonians

$$\begin{aligned} H_1 &= \frac{1}{2}E_c\,(\sigma_z^{(1)} + \sigma_z^{(2)}) - \frac{1}{2}E_J\,(\sigma_x^{(1)} + \sigma_x^{(2)}) \\ H_2 &= -\frac{1}{2}E_J\,(\sigma_x^{(1)} + \sigma_x^{(2)}) - \frac{1}{2}E_L\,\sigma_y^{(1)}\sigma_y^{(2)} \\ H_3 &= \frac{1}{2}E_c\,\sigma_z^{(2)} - \frac{1}{2}E_J\,(\sigma_x^{(1)} + \sigma_x^{(2)}) \\ H_4 &= \frac{1}{2}E_c\,\sigma_z^{(1)} - \frac{1}{2}E_J\,(\sigma_x^{(1)} + \sigma_x^{(2)}) \end{aligned} \tag{11}$$

and all basic gates are simulated in 15 steps (9).

The efficiency of our simulation is defined by a test function, f_{test}. It is a function of 15 time variables

$$f_{test}(t_1,\ldots,t_{15}) = \sum_{i,j=1}^{4} |(U_{\text{gate}})_{ij} - (U(t_1,\ldots,t_{15}))_{ij}|^2 = ||U_{\text{gate}} - U||^2 \tag{12}$$

In our numerical simulations f_{test} usually attains values of 10^{-8} or less. Taking into account more decimal digits in the approximation of the time parameters results to a further decrease of its value. Gates that have been tested numerically are all the important two-qubit gates for quantum computation such as the CNOT gate, the SWAP gate, the Quantum Fourier Transform gate for two qubits, several controlled gates etc. The ratio of the values of the external switches tuning amplitudes over the magnitude of the interaction is not in the area of hard pulses. Numerical results are available upon request from the authors.

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